Journal of Applied Mathematics and Mechanics 73 (2009) 574–580

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

An asymptotic analysis of the steady process of saturation of a laminated porous material☆

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ARTICLE INFO

Article history:
Received 8 December 2008

ABSTRACT

The propagation of a steady saturation front in a double-layer porous material, situated between impenetrable walls, is investigated. The closed system of equations and boundary conditions are written on the assumption that the displacing and displaced phases, the viscosities of which differ considerably, are connected, and that the capillary pressure on the interface is constant. The features of the behaviour of the interface in the neighbourhood of the boundary of the layers are investigated in the case when the layer thicknesses differ considerably. When the permeabilities of the layers differ considerably, asymptotic expressions are obtained for the pressure and shape of the interface, and a comparison is made with the results of a numerical solution of the complete problem and with the known asymptotic relations obtained when using a simplified boundary condition at the interface.

In the production of composite materials, the technology of the vacuum infusion of a resin into closed moulds is widely employed, which, from a mechanical point of view, reduces to seepage saturation of a porous material with a viscous fluid. To accelerate the process of saturation a thin distributed layer of high-permeability material is often added to the porous material – a relatively thick layer of low-permeability material.[1] As a result, the problem arises of the evolution of the interface when the double-layer porous material is being saturated with the viscous fluid, when the thicknesses of the layers and their permeabilities differ considerably.

The problem is also of interest from the point of view of possible geophysical applications, namely, when analysing the processes of the combined motion of water and air in laminated natural strata.

1. Formulation of the problem

A porous sample, saturated with fluid, consists of two layers with different properties, contained between Impermeable walls (Fig. 1). The left end of the sample is connected to a reservoir of viscous fluid, and the right end is connected to an air chamber. By reducing the pressure in the chamber, a pressure drop is obtained in the liquid, which also gives rise to a process of seepage saturation of the porous sample. While the external conditions are unchanged, a configuration of the interface is established and the whole of its evolution reduces to simple motion at a fixed rate. This steady configuration is also the one required.

We will direct the $\hat{y}$ axis of a Cartesian system of coordinates perpendicular to the interface plane, and we will choose the $\hat{x}$ axis in the direction of motion of the interface. We will connect the origin of coordinates with a certain point C on the interface, which moves along the boundary between the layers. We will assume that the parameters do not change in the direction of the normal to the $\hat{x}$, $\hat{y}$ plane. The problem is completely stationary in this system of coordinates.

The characteristics of the upper layer will be denoted by a superscript plus, and those of the lower layer will be denoted by a superscript minus. The layers have a thickness $h^{\pm}$, a porosity $m^{\pm}$ and a permeability $k^{\pm}$, respectively. The overall thickness of the sample is much less than its length in the $\hat{x}$ direction, so that the length can be assumed to be infinite. We will denote the unknown velocity of motion of the interface by $\hat{u}$ (everywhere we will take the velocities to mean the velocities of motion with respect to the porous skeleton).

The interface divides the sample into two zones: unwetted, to the right of the interface, and completed wetted $\Omega^{\pm}$ to the left of the interface. Since the viscosity of the air is much less than the viscosity of the liquid displacing it, the pressure in the unwetted zone will be the same everywhere (we will take it to be zero). In the wetted part of the sample the fluid is incompressible, and its motion obeys Darcy's

☆ *Pril. Mat. Mekh.* Vol. 73, No. 5, pp. 799-808, 2009.
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doi:10.1016/j.jappmathmech.2009.11.010

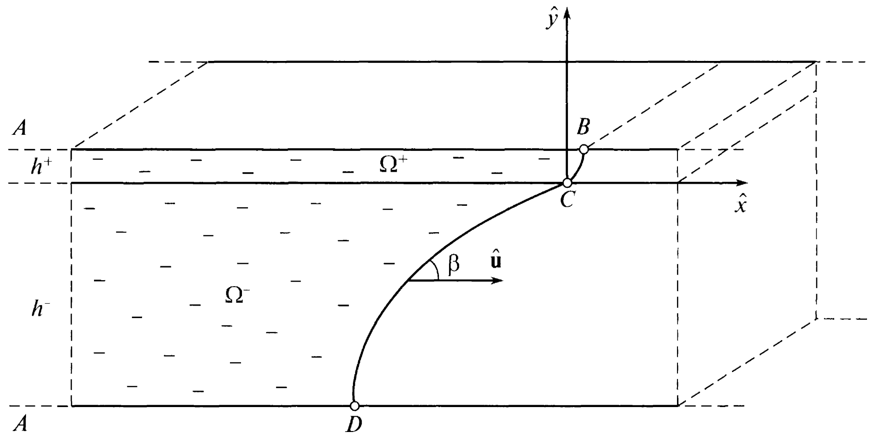

Fig. 1.

law[2,3]

$$\Omega^{\pm}: \quad \nabla \mathbf{v}^{\pm} = 0, \quad \mathbf{v}^{\pm} = -\frac{k^{\pm}}{\mu}\nabla p^{\pm}$$

where μ is the viscosity of the fluid, $v^{\pm}(\hat{x}, \hat{y})$ are the rates of seepage, $k^{\pm}$ is the permeability of the layers and $p^{\pm}(\hat{x}, \hat{y})$ are the pressure distributions in the wetted part of the layers.

The motion of the interface in the steady process of the saturation of a double-layer porous material is represented by the boundary-value problem[4]

$$\Omega^{\pm}: \quad \frac{\partial^2 p^{\pm}}{\partial \hat{x}^2} + \frac{\partial^2 p^{\pm}}{\partial \hat{y}^2} = 0$$

$$AB \cup AD: \quad \frac{\partial p^{\pm}}{\partial \hat{y}} = 0; \quad AC: \quad p^{+}\Big|_{\hat{y}\to+0} = p^{-}\Big|_{\hat{y}\to-0}, \quad k^{+}\frac{\partial p^{+}}{\partial \hat{y}}\Big|_{\hat{y}\to+0} = k^{-}\frac{\partial p^{-}}{\partial \hat{y}}\Big|_{\hat{y}\to-0}$$

$$BC \cup CD: \quad p^{\pm} = 0, \quad \frac{\partial p^{\pm}}{\partial \hat{y}} = \frac{\mu m^{\pm}\hat{u}}{2k^{\pm}}\sin 2\beta; \quad A: \frac{\partial p^{\pm}}{\partial \hat{x}} = -\lambda^2 \tag{1.1}$$

We will briefly explain the boundary conditions. The substrate of the lower layer AD and the topping of the upper layer AB are impermeable. The following boundary conditions are satisfied at the interface between the layers AC: continuity of the pressures and of the normal components of the seepage flow of fluid. The interface $BC \cup CD$ is free, and two boundary conditions are specified on it, namely, dynamic, which means that the capillary forces are constant along the whole interface, and kinematic, which means that the free boundary is formed by material particles of fluid, the velocity of which in each layer is defined as $v^{\pm}/m^{\pm}$. To write the last condition we have chosen one of the possible forms given in Refs. 2,3 and 5. We denote by β the angle of inclination to the horizontal of the tangent to the free boundary (see Fig. 1).

The condition at infinity on the left closes the problem. Since the porous sample is bounded above and below by impermeable planes, and backword flow between the layers is allowed, the pressure across the layers, as a source of perturbations, becomes equalized as one moves away from the interface.[6] Correspondingly, the condition at infinity on the left can be written in the form of a pressure gradient, specified over the whole thickness of the sample.

Note that the velocity $\hat{u}$ of motion of the interface is established from the balance relations[4]

$$\hat{u} = \frac{\lambda^2}{\mu}\frac{k^{+}h^{+} + k^{-}h^{-}}{m^{+}h^{+} + m^{-}h^{-}}$$

2. The case when one of the layers is thin

To fix our ideas, we will assume that the upper layer is a thin layer. In this case, the thickness of the lower layer h^{-} is the characteristic dimension of the problem, and we can change to the dimensionless coordinates $x = \hat{x}/h^{-}, \quad y = \hat{y}/h^{-}$. The dimensionless thickness of the upper layer becomes the small parameter

$$\alpha = h^{+}/h^{-} \ll 1 \tag{2.1}$$

and problem (1.1) can be simplified considerably. For this purpose we will consider the following function as the auxiliary harmonic function in the region Ω^{+}

$$g(x,y) \equiv \partial p^{+}/\partial y \tag{2.2}$$

At the boundaries of the upper layer it satisfies the conditions

$$y \to +0: \quad g(x,y) = G(x); \quad y = \alpha: \quad g(x,y) = 0 \tag{2.3}$$

where we have used the matching conditions of problem (1.1) and introduced the notation

$$G(x) \equiv \left(k^-/k^+\right)\partial p^-/\partial y\big|_{y\to -0} \tag{2.4}$$

We will estimate the derivative $\partial g/\partial y$ at the boundary of the layers $y = +0$. Using an expansion of the function $g(x,y)$ in a Taylor series in the variable y in the neighbourhood $y = +0$ and the second boundary condition of (2.3), we obtain when $y=\alpha \ll 1$

$$\partial g/\partial y\big|_{y\to +0} = -G(x)/\alpha + O\left(\left|\partial^2 g/\partial x^2\right|\alpha\right) \tag{2.5}$$

where the remaining term is converted taking into account the harmonic form of the function g.

To estimate the order of the quantity $|\partial^2 g/\partial x^2|$ we will use the already mentioned statement on the equalization of the pressure across the layers for the type of seepage flows considered. It is well known,[6] that this equalization occurs at a distance of the order of the thickness of the porous sample from all sources of perturbation of the uniform field ∇p^+. Hence, whereas in the transverse section x of the sample, by conditions (2.3), we have $O(|g(x,y)|) = O(|G(x)|)$, in the transverse section x - L, where

$$L \approx \begin{cases} 1, & x \le x_D \\ 1 + |x - x_D|, & x_D < x < 0 \end{cases}$$

the function $g(x,y)$ and all its partial derivatives are negligibly small compared with $|G(x)|$. Consequently, the following estimate holds

$$O\left(\left|\frac{\partial^2 g}{\partial x^2}\right|\right) = O\left(\left|\frac{\partial g}{\partial x}\right|\right) = O(|g(x,y)|) = O(|G(x)|)$$

using which, we obtain from relation (2.5) an expression for the derivative $\partial g/\partial y$ on the boundary of the layers. Changing to functions of the pressure $p^{\pm}(x.y)$ using formulae (2.2) and (2.4), we obtain

$$\left.\frac{\partial^2 p^+}{\partial y^2}\right|_{y\to +0} = -\varepsilon \left.\frac{\partial p^-}{\partial y}\right|_{y\to -0}\left[1 + O\left(\alpha^2\right)\right]; \quad \varepsilon = \frac{k^- h^-}{k^+ h^+} \tag{2.6}$$

In turn, we obtain the following relation from the matching conditions of problem (1.1)

$$\left.\frac{\partial^2 p^+}{\partial x^2}\right|_{y\to +0} = \left.\frac{\partial^2 p^-}{\partial x^2}\right|_{y\to -0} \tag{2.7}$$

The function $p^+(x,y)$ satisfies Laplace's equation everywhere in the region Ω^+. We will write it on the line $y = \text{const} \in (0, \alpha)$. We then take the limit const$\to$+0 and use formula (2.6) and (2.7) to change to the function $p^-(x,y)$, already defined in the region Ω^-. As a result we obtain the boundary condition for this function on the boundary of the regions Ω^- and Ω^+ (Ref. 4)

$$y \to -0: \quad \frac{\partial^2 p^-}{\partial x^2} - \varepsilon \frac{\partial p^-}{\partial y}\left[1 + O\left(\alpha^2\right)\right] = 0 \tag{2.8}$$

representing the influx of fluid into the lower layer from the thin upper layer.

Unlike the approach described earlier[4], the derivation of boundary relation (2.8) emphasises that the unique limitation for using it as the boundary condition for the function $p^-(x,y)$, specified in the lower layer, is the thickness of the upper layer, i.e., $\alpha \ll 1$. At the same time, the parameter ε, representing the ratio of the permeable properties of the layers, can take any values including very high values.

For a complete change to a dimensionless formulation of the problem we take as the characteristic velocity the velocity $\hat{u}_*$ of the material of particles of the fluid at infinity in the lower layer

$$\hat{u}_* = \frac{\lambda^2 k^-}{\mu m^-}$$

Then the dimensionless velocity of motion of the interface will be given by the formula

$$u = \frac{\hat{u}}{\hat{u}_*} = \frac{1 + \varepsilon^{-1}}{1 + \delta}; \quad \delta = \frac{m^+ h^+}{m^- h^-} \tag{2.9}$$

The dimensionless complex δ obviously characterises the ratio of the capacitive properties of the layers.

Instead of the pressure distribution function $p^-(x,y)$ in the region Ω^- we will introduce the dimensionless velocity of the slow potential

$$\varphi(x,y) = -\frac{1}{\lambda^2 h^-} p^-(x,y)$$

so that $\nabla\varphi$ is the dimensionless flow velocity of the fluid in the region Ω^-.

As a result we can write a dimensionless formulation of the boundary-value problem of the steady process of saturation of a double-layer porous material in the case of a thin upper layer[4]

$$\begin{aligned}&\Omega^{-}: \quad \Delta\varphi = 0\\&AD: \quad \frac{\partial\varphi}{\partial y} = 0; \quad AC: \quad \frac{\partial^2\varphi}{\partial x^2} - \varepsilon\frac{\partial\varphi}{\partial y} = 0\\&CD: \quad \varphi = 0, \quad \frac{\partial\varphi}{\partial y} = -\frac{u}{2}\sin 2\beta; \quad A: \quad \frac{\partial\varphi}{\partial x} = 1\end{aligned} \tag{2.10}$$

The configuration of the interface is defined by the equation $\varphi = 0$. If we formally represent it by means of the function f(x)

$$CD: \quad y = f(x) \tag{2.11}$$

the relation with the value of the angle β

$$\mathrm{tg}\beta = f'(x) \tag{2.12}$$

will be the defining one for this function.

By virtue of assumption (2.1) the parameter δ is small, at least $\delta < 1$. The product of the parameters ε and δ

$$\varepsilon\delta = \frac{k^{-}}{m^{-}}\left(\frac{k^{+}}{m^{+}}\right)^{-1}$$

defines in which layer the velocity of the material particles of fluid will be greater at infinity: in the first layer ($\varepsilon\delta \leq 1$ and then $\beta \leq \pi/2$ everywhere on DC) or in the lower layer ($\varepsilon\delta \geq 1$ and then $\beta \geq \pi/2$ everywhere on DC). The case $\varepsilon\delta \leq 1$ is of more interest, and in order to avoid lengthy formulae we will henceforth confine ourselves to this case.

An analysis of problem (2.10) enables us to establish directly the main characteristics of the flow in the neighbourhood of the point C. It is only necessary to introduce the stream function $\psi(x,y)$, harmonically conjugate to the potential $\varphi(x,y)$. Then, taking the boundary AD as the zeroth streamline, we can determine, from balance considerations, the values of the stream function at all characteristic points of the region Ω^{-} (Ref.4)

$$AD: \quad \psi = 0; \quad AC: \quad \psi|_A = 1, \quad \psi|_C = u$$

Using the Cauchy–Riemann relations, we replace $-\partial\varphi/\partial y$ by $\partial\psi/\partial x$ in the boundary condition on the section AC and, integrating the result along the boundary from $-\infty$ to the current value of x, we obtain another form of expressing this boundary condition

$$AC: \quad \frac{\partial\varphi}{\partial x} - 1 + \varepsilon(\psi - 1) = 0$$

Hence, taking into account the known value of the stream function at the point C we can determine the component $-\partial\varphi/\partial x$ of the flow velocity at this point

$$\left.\frac{\partial\varphi}{\partial x}\right|_C = \varepsilon\delta u \tag{2.13}$$

Further, using another form of writing the kinematic boundary condition on the interface CD (Ref.3)

$$CD: \quad \varphi = 0, \quad \left(\frac{\partial\varphi}{\partial x} - \frac{u}{2}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 = \frac{u^2}{4}$$

we can also determine the component $\partial\varphi/\partial y$ of the flow velocity at the point C

$$\left.\frac{\partial\varphi}{\partial y}\right|_C = -\sqrt{\varepsilon\delta(1 - \varepsilon\delta)}\, u \tag{2.14}$$

Knowing completely the flow velocity at the point C and taking into account the equipotential nature of the boundary CD, we can also obtain the angle which the boundary CD makes with the boundary AC (see Fig. 1):

$$\beta_C = \arcsin\sqrt{\varepsilon\delta}$$

We have thus established the fundamental characteristics of the flow in the neighbourhood of the point C. We add that, by differentiating the boundary conditions on the parts AC and DC along these parts, we can also obtain more detailed characteristics of the flow at the point C, for example, the second derivatives of the function φ with respect to the variables x and y, the curvature of the interface etc.

3. The case of a high-permeability thin layer

In this case, the single small parameter $\varepsilon \ll 1$ is once again the most interesting for applications. It enables us to make use of asymptotic methods of analysing boundary-value problem (2.10). In particular, such an analysis has been trased on replacing the boundary condition on section AC by another:[4]

$$AC: \quad \partial\varphi/\partial x = 1 \tag{3.1}$$

which results from the existing condition when $\varepsilon = 0$. The neighbourhood of the point C is then, obviously, not covered, and the zone of action of the asymptotic expansion will be part of the region Ω^{-1}, adjoining infinity. The solution of the problem obtained was constructed analytically, and it turned out that the best approximation of the configuration of the interface is a straight line, the slope of which to the horizontal is determined solely by the quantity **u**:

$$\beta_0 = \operatorname{arctg}\frac{1}{\sqrt{u-1}} \approx \sqrt{\varepsilon(1+\delta)}$$

(the estimate is given taking into account formula (2.9)). Correspondingly, the length of the interface along the horizontal was also estimated to be

$$x_D \approx -\frac{1}{\sqrt{\varepsilon(1+\delta)}} \tag{3.2}$$

At the same time, analysis showed that the solution of problem (2.10) when the boundary condition on the part AC is replaced by condition (3.1) gives the principal term of the asymptotic expansion of the solution of the initial problem in powers of $\sqrt{\varepsilon}$, rather than in powers of ε Hence, we can expect that the accuracy of the estimates obtained previously in Ref.4 will be low.

It is difficult to refine the estimates by constructing the next term of the expansion. Instead we propose to construct another asymptotic expansion. For this purpose we will introduce new coordinates (X,Y) and a new notation $\Phi(X,Y)$ for the potential φ as a function of these coordinates

$$X = x\sqrt{\varepsilon}, \quad Y = y; \quad \Phi(X,Y) \equiv \varphi(x,y)\big|_{x = X/\sqrt{\varepsilon},\, y = Y} \tag{3.3}$$

and also new notation for the function F(X), which gives representation (2.11) of the interface DC in the new coordinates

$$DC: \quad Y = F(X), \quad F(X) \equiv f(x)\big|_{x = X/\sqrt{\varepsilon}} \tag{3.4}$$

In view of expression (2.12) the latter is related to the angle β by the formula

$$\operatorname{tg}\beta = \sqrt{\varepsilon}\left[F_0'(X) + \varepsilon F_1'(X) + O(\varepsilon^2)\right] \tag{3.5}$$

Both unknown functions Φ(X Y) and F(X) will be sought in the form of an asymptotic expansion in powers of ε

$$\Phi(X,Y) = \Phi_0(X,Y) + \varepsilon\Phi_1(X,Y) + O(\varepsilon)^2$$
$$F(X) = F_0(X) + \varepsilon F_1(X) + O(\varepsilon)^2 \tag{3.6}$$

We substitute the new representation of solution (3.3) - (3.6) into the equation and boundary conditions of problem (2.10), with the exception of the condition on the section AD and the condition at infinity. Unlike the expansion proposed earlier,[4] the zone of action of this asymptotic expansion will be the neighbourhood of the point C, where the transition to the new coordinate X by compressing the old coordinate x enables us to hope that a considerable part of the interface will fall in this neighbourhood.[7]

We will only follow the principal term of the asymptotic expansion in ε. Laplace's equation takes the form $\partial^2\Phi_0/\partial Y^2 = 0$. Hence it follows that the function Φ_0(X,Y) can be represented in the form

$$\Phi_0(X,Y) = A(X)Y + B(X) \tag{3.7}$$

It follows from the boundary condition on the section AC

$$AC: \quad \partial^2\Phi_0/\partial X^2 - \partial\Phi_0/\partial Y = 0$$

that A(X)=B″(X) in representation (3.7). As a result the function Φ_0(X,Y) can be expressed solely in terms of the function B(X)

$$\Phi_0(X,Y) = B''(X)Y + B(X) \tag{3.8}$$

The first boundary condition on the interface now takes the form

$$DC: \quad \Phi_0(X,Y)\big|_{Y = F_0(X)} = 0$$

and, taking into account the known form (3.8) of the function Φ_0(X,Y), we can express the unknown function F_0(X) in terms of the same function B(X)

$$F_0(X) = -B(X)/B''(X) \tag{3.9}$$

Hence, the problem has been reduced to finding the single unknown function B(X). To determine it we have the second boundary condition on the interface. In X,Y coordinates it takes the form

$$DC: \quad \partial\Phi_0/\partial Y|_{Y=F_0(X)} = -u\,\mathrm{tg}\beta\big/\left(1+\mathrm{tg}^2\beta\right)$$

Substituting expressions (3.5) and (3.8) here, we obtain the relation

$$B''(X) = -\sqrt{\varepsilon}uF_0'(X)$$

Integrating it with respect to X, taking into account the obvious condition $F_0(0)=0$, and substituting expression (3.9) into the result, we obtain an ordinary differential equation in the function B(X)

$$\left[B'(X)-B'(0)\right]B''(X)-\sqrt{\varepsilon}uB(X)=0 \tag{3.10}$$

The quantity $B'(0)$ can be calculated using the known component $\partial\varphi/\partial x$ of the flow velocity at the point C. In fact, in terms of the function $\Phi_0(X,Y)$ expression (2.13) takes the form

$$\partial\Phi_0/\partial X|_{X=Y=0} = \sqrt{\varepsilon}\delta u$$

whence, taking representation (3.8) of the function $\Phi_0(X,Y)$ into account, it follows that

$$B'(0) = \sqrt{\varepsilon}\delta u \tag{3.11}$$

This is the first initial condition for ordinary differential equation (3.10). The second will be the condition

$$B(0)=0 \tag{3.12}$$

which follows from the form (3.9) of the function $F_0(X)$ and the condition $F_0(0)=0$.

Hence, we must integrate the second-order non-linear ordinary differential equation (3.10) with initial conditions (3.11) and (3.12). It can be shown that the solution of this equation, which satisfies initial condition (3.12), will be a polynomial of the following form

$$B(X) = u\sqrt{\varepsilon}\left(b_1X+b_2X^2+b_3X^3\right) \tag{3.13}$$

In fact, calculating the derivatives of this polynomial and substituting them into Eq. (3.10), we obtain, after cancelling the factor $u^2\varepsilon$, the relation

$$\left[4b_2^2-b_1\right]X+\left[b_2(18b_3-1)\right]X^2+\left[b_3(18b_3-1)\right]X^3=0$$

Equating the coefficients of different powers of X to zero, we arrive at a system of three equations in the three parameters b_1, b_2 and b_3. If we assume that $b_3=0$, then only trivial solutions $B(X)\equiv 0$ are possible, which makes no physical sense. If we assume that $b_3\neq 0$, the system degenerates, since we obtain the same equations $18b_3 - 1 = 0$ for the coefficients of X^2 and X^3. As a result we will have

$$b_3 = 1/18, \quad b_1 = 4b_2^2 \tag{3.14}$$

Consequently, polynomial (3.13) will in fact be the solution of Eq. (3.10), when its parameters b_1 and b_3 have the form (3.14). The value of the last parameter b_2 gives the remaining initial condition (3.11)

$$b_2 = -\sqrt{\delta}/2$$

The minus sign is chosen from the considerations that the vertical component of the flow velocity $\partial\Phi/\partial y$ at the point C must be negative. As a result we have

$$B(X) = u\sqrt{\varepsilon}\left(\delta X-\frac{\sqrt{\delta}}{2}X^2+\frac{X^3}{18}\right)$$

and, correspondingly, the form of the required functions is determined, namely,

$$\frac{1}{u\sqrt{\varepsilon}}\Phi_0(X,Y) = \left(-\sqrt{\delta}+\frac{X}{3}\right)Y+\left(\delta X-\frac{\sqrt{\delta}}{2}X^2+\frac{X^3}{18}\right)$$

$$F_0(X) = \frac{3}{2}\delta-\frac{1}{6}\left(X-3\sqrt{\delta}\right)^2 = X\left(\sqrt{\delta}-\frac{X}{6}\right)$$

4. Analysis of the results

We return to the initial coordinates x, y and the stream potential $\varphi(x,y)$ (see formulae (3.3)) and we obtain the principal term of the asymptotic expansion of the functions $\varphi(x,y)$ and f(x)

$$\frac{1}{u\varepsilon}\varphi_0(x,y) = \left(-\sqrt{\frac{\delta}{\varepsilon}}+\frac{x}{3}\right)y+\left(\delta x-\frac{\sqrt{\delta\varepsilon}}{2}x^2+\varepsilon\frac{x^3}{18}\right) \tag{4.1}$$

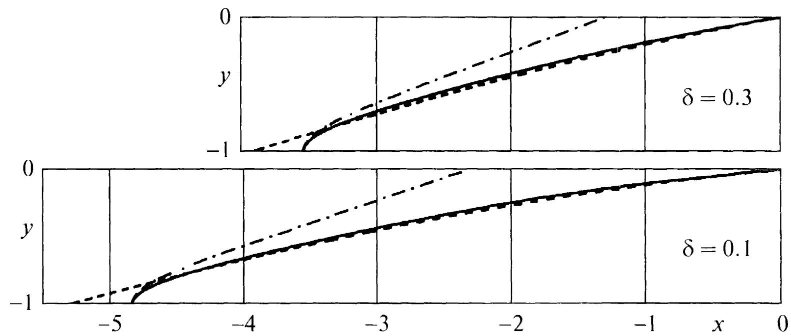

Fig. 2.

$$f_0(x) = x\sqrt{\varepsilon}\left(\sqrt{\delta} - \frac{x}{6}\sqrt{\varepsilon}\right) \quad (4.2)$$

The partial derivatives of the function $\varphi_0(x,y)$ at the point C

$$\left.\frac{\partial\varphi_0}{\partial x}\right|_C = \varepsilon\delta u, \quad \left.\frac{\partial\varphi_0}{\partial y}\right|_C = -\sqrt{\varepsilon\delta}u \quad (4.3)$$

are identical with the exact values (2.13) and (2.14) when the principal term of the asymptotic expansion in powers of ε is taken into account.

The equation $y = f_0(x)$, which defines the shape of the interface DC, will obviously be the equation of an inverted parabola, the vertex of which is shifted by an amount $3\sqrt{\delta/\varepsilon}$ along the x axis and by amount $3\delta/2$ along the y axis. The length of the interface along the horizontal is characterized by the coordinate x_D, which can be found by putting $f_0(x) = -1$:

$$x_D = \frac{3}{\sqrt{\varepsilon}}\left(\sqrt{\delta} - \sqrt{\delta + \frac{2}{3}}\right) = -\sqrt{\frac{6}{\varepsilon}}\left[1 - \sqrt{\frac{3\delta}{2}} + \frac{3\delta}{4} + O(\delta^2)\right] \quad (4.4)$$

An estimate is obtained for small δ.

Comparison of the estimates (4.3) and (3.2) shows that both versions of the asymptotic analysis give the same scale of the length of the interface along the horizontal $|x_D| \sim 1\sqrt{\varepsilon}$, but the estimates differ by a factor of $\sqrt{6}$ (if we neglect quantities $O(\delta)$).

An answer to the question of which of the estimates is more accurate can be obtained by a computational experiment. For problem (2.10) we constructed an iterational method, similar to the Levi-Civita method in the theory of jets of an ideal fluid,[8] with separation of the fractional-power singularities in the solution and a determination of their moments by internal iterations. On the basis of this, we were able to obtain, with satisfactory accuracy, a numerical solution of the problem for $\varepsilon \geq 0.1$ and $\delta \geq 0.1$. In Fig. 2 we show the configurations of the interface DC, obtained by numerical calculations (the continuous line), and constructed by asymptotic analysis[4] (the dash-dot line) and by asymptotic formula (4.2) (the dashed line) when $\varepsilon = 0.1$ and $\delta = 0.3$ (the upper part of Fig. 2) and $\delta = 0.1$ (the lower part). It can be seen that the asymptotic formula (4.2) describes the shape of the interface DC much more accurately; thus, the error in determining its length along the horizontal is approximately 10% for both versions of the calculation, i.e. this quantity is of the order of ε.

For the curves constructed from the results of the asymptotic form, obtained previously,[4] this error amounts to approximately 35 - 50%, i.e. it will be a quantity of the order of $\sqrt{\varepsilon}$. The error in determining the radius of curvature of the interface at the point D is of the same order.

Hence, asymptotic expansion (4.1), (4.2) of the solution of problem (2.10) satisfactorily describes the configuration of the interface DC for $\varepsilon = 0.1$ down to a small neighbourhood of the point D. Note also that, using a non-orthogonal expansion of the coordinates, similar to that described earlier,[9] the results obtained can be extended to the case of extremely anisotropic layers.

Acknowledgements

I wish to thank A. G. Yegorov and J. G. Kornev for useful discussions.

This research was financed by the Russian Foundation for Basic Research (08-01-00548).

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Translated by R.C.G.